Alternating Minimization for Regression with Tropical Rational Functions

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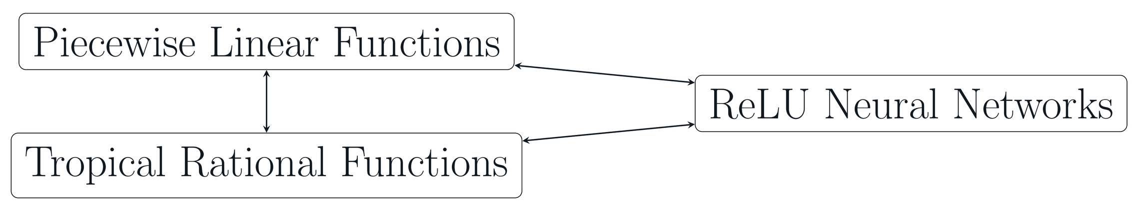
Introduction

Recent work [3] has connected piecewise linear functions, fully connected ReLU networks, and **Tropical Algebra**.



Motivated by these connections, we address the following:

Question

How can we leverage tropical **algebraic structure** to solve regression problems directly over the space of tropical rational functions?

Tropical Algebra Background

Tropical algebra is algebra using the max-plus semiring

$$\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot), \qquad a \oplus b := \max(a,b), \qquad a \odot b := a + b.$$

A **tropical rational function** is a tropical quotient of two tropical polynomials:

$$r(\mathbf{x}) = p(\mathbf{x}) \oslash q(\mathbf{x}) = \max_{j \in [D]} (p_j + \langle \mathbf{w}^{(j)}, \mathbf{x} \rangle) - \max_{j \in [D]} (q_j + \langle \mathbf{w}^{(j)}, \mathbf{x} \rangle).$$

Such functions are piecewise linear and the difference of convex functions.
Tropical linear algebra gives the solution to minimum residual linear problems: Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Denote by $\boxplus$ and $\boxplus'$ the max-plus and min-plus matrix-vector products, respectively.

$$\arg\min_{\mathbf{x}} \|A \boxplus \mathbf{x} - \mathbf{b}\|_\infty = ((-A^\top) \boxplus' \mathbf{b}) + \frac{1}{2} \left\| A \boxplus \left((-A^\top) \boxplus' \mathbf{b} \right) - \mathbf{b} \right\|_\infty \qquad (1)$$

Key Point

The solution to problem (1) can be found using only **matrix-vector products** and **addition**. This has been leveraged to quickly solve **tropical polynomial regression** (see e.g. [2]).

Regression with Tropical Rational Functions

Given:

- Dataset: $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^N \subseteq \mathbb{R}^n \times \mathbb{R}$
- Permissible Exponents: $W = \{\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(D)}\} \subseteq \mathbb{Z}^n_{\geq 0}$.

Set $X \in \mathbb{R}^{N \times D}$ to have entries $X_{i,j} = \langle \mathbf{w}^{(j)}, \mathbf{x}^{(i)} \rangle$.

Problem Formulation

With the above notation, we wish to find

$$\arg\min_{\mathbf{p},\mathbf{q}} \|X \boxplus \mathbf{p} - X \boxplus \mathbf{q} - \mathbf{y}\|_\infty \qquad (2)$$

Alternating Minimization

For fixed $\mathbf{q}$, $\min_{\mathbf{p}} \|X \boxplus \mathbf{p} - (X \boxplus \mathbf{q} + \mathbf{y})\|_\infty$ is a tropical polynomial regression problem. So, we can compute

$$\mathbf{p}^*(\mathbf{q}) = \arg\min_{\mathbf{p}} \|X \boxplus \mathbf{p} - (X \boxplus \mathbf{q} + \mathbf{y})\|_\infty$$

By (1), this can be done quickly. Similarly, for fixed $\mathbf{p}$ we compute $\mathbf{q}^*(\mathbf{p})$ using (1).

Heuristic

Initialize $\mathbf{p}^0$ and $\mathbf{q}^0$ then iteratively set $\mathbf{p}^k \leftarrow \mathbf{p}^*(\mathbf{q}^{k-1})$ and $\mathbf{q}^k \leftarrow \mathbf{q}^*(\mathbf{p}^k)$

Results

1. Univariate Data: $x^{(i)} \in [-1, 12]$ uniformly spaced, $y^{(i)} = \sin(x^{(i)}) + \epsilon^{(i)}$, $N = 200$, $W = \{0, 1, \dots, 15\}$

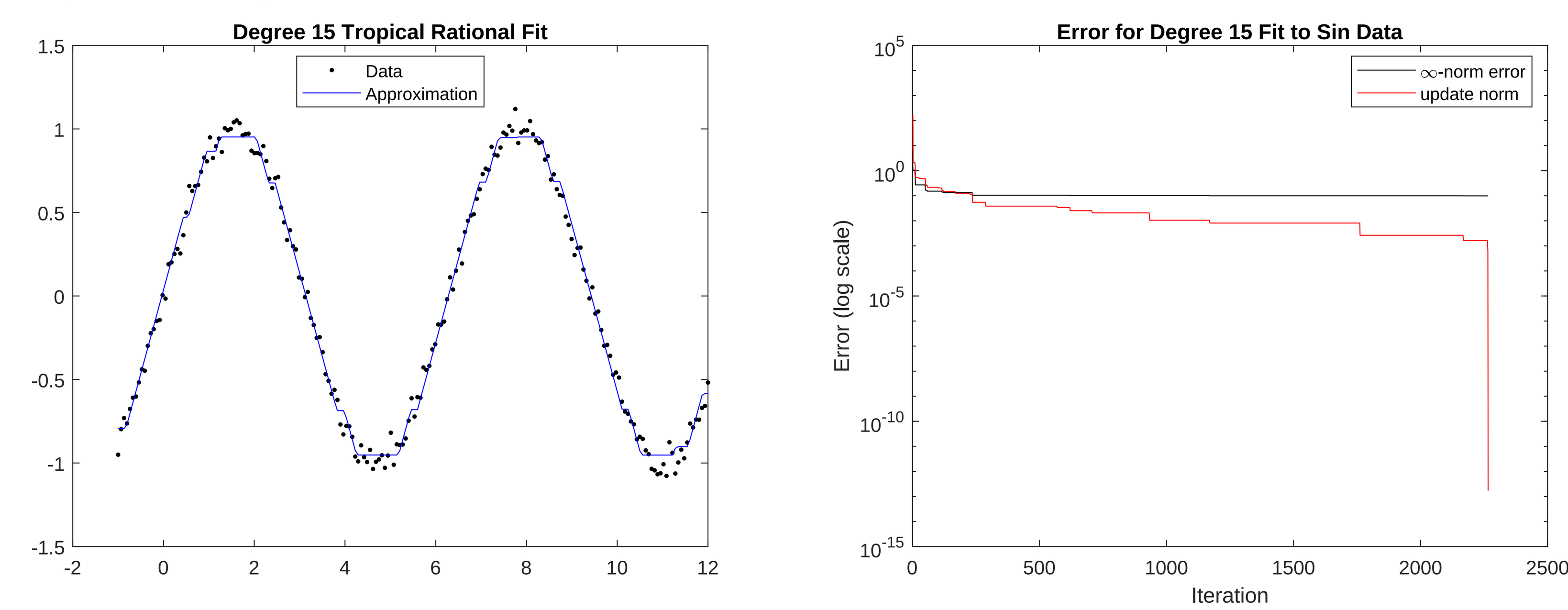


Fig. 1: Approximation (left) and Convergence History (right) for regression on sin dataset

2. Bivariate Data: $(\mathbf{x}, y) = \texttt{peaks}$, $\quad N = 49^2$, $\quad W = \{0, \dots, 31\}^2$

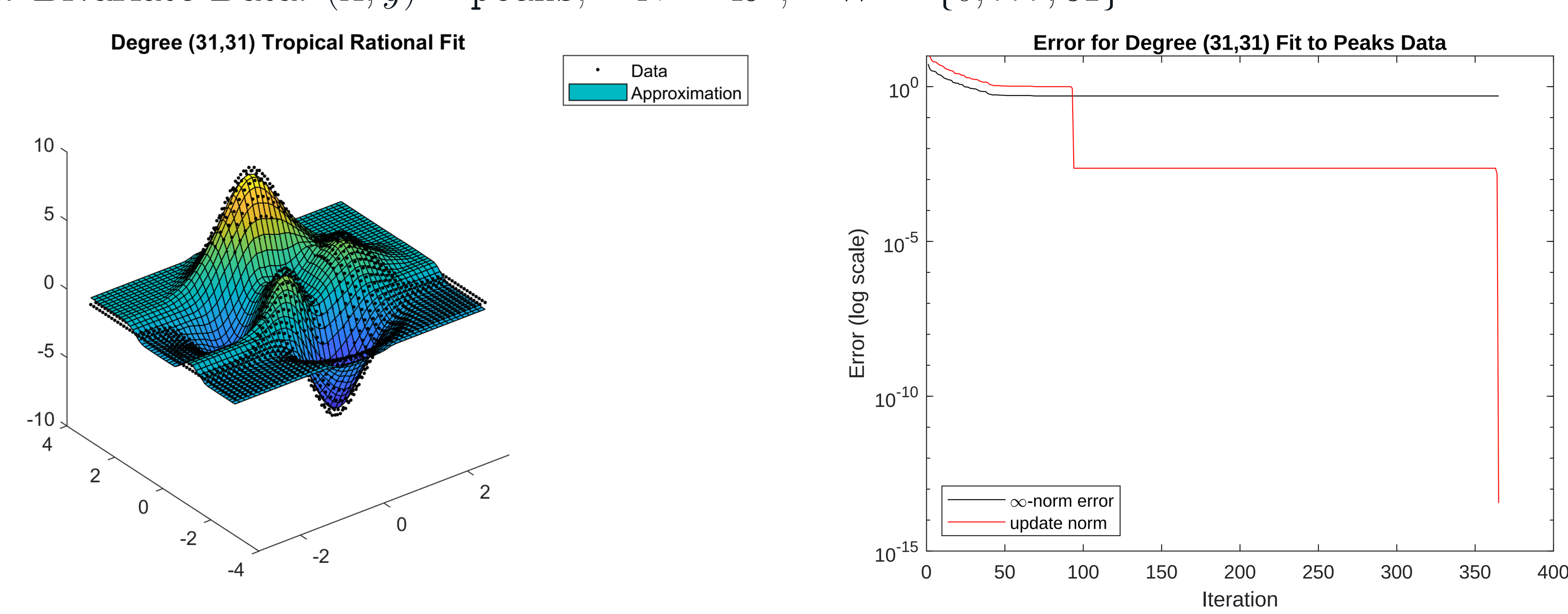


Fig. 2: Approximation (left) and Convergence History (right) for regression on peaks dataset

3. Data Generated From Tropical Rational Functions: Average training and validation error across five trials on data generated from 6 variable tropical rational functions.
$N = 10,000$, $\quad \mathbf{x}^{(i)} \sim \text{Uniform}([-5,5]^6)$.

Degree	1	2	3	4	5
Relative Training Error	2.372×10^{-15}	5.869×10^{-15}	9.108×10^{-15}	1.286×10^{-14}	9.373×10^{-6}
Relative Validation Error	0.1271	0.2019	0.2869	0.3631	0.3598

Geometry of the Loss Function

Three observations:

- The loss $\mathcal{L}(\mathbf{p}, \mathbf{q}) = \|X \boxplus \mathbf{p} - X \boxplus \mathbf{q} - \mathbf{y}\|_\infty$ is a tropical rational function of the coefficients p_j, q_j.
- There is always an optimizer $(\mathbf{p}^*, \mathbf{q}^*)$ where $\nabla \mathcal{L}(\mathbf{p}^*, \mathbf{q}^*)$ does not exist.
- The alternating minimization method produces iterates where $\mathcal{L}$ is nondifferentiable.

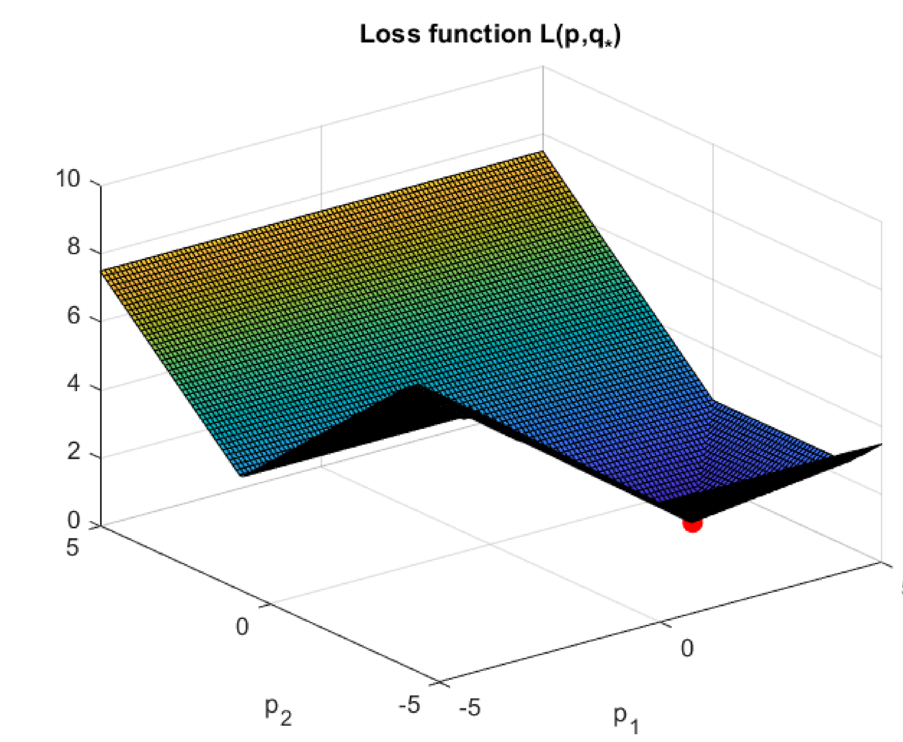


Fig. 3: Example loss function $\mathcal{L}(\mathbf{p}, \mathbf{q}^*)$ where the optimal q^* is known.

Get in Touch

Read our paper [1]:

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References

[1] Alex Dunbar and Lars Ruthotto. "Alternating minimization for regression with tropical rational functions". In: *Algebr. Stat.* 15.1 (2024), pp. 85–111. ISSN: 2693-2997,2693-3004.

[2] Petros Maragos and Emmanouil Theodosis. "Multivariate tropical regression and piecewise-linear surface fitting". In: *ICASSP 2020-2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE. 2020, pp. 3822–3826.

[3] Liwen Zhang, Gregory Naitzat, and Lek-Heng Lim. "Tropical geometry of deep neural networks". In: *International Conference on Machine Learning*. PMLR. 2018, pp. 5824–5832.

Acknowledgement

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